Journal of Engineering Mathematics **31:** 43–57, 1997.
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A mathematical model and its analytical solution for slow depressurization of a gas-filled vessel

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Received 2 June 1994; accepted in revised form 27 January 1995

Abstract. The operation safety of depressurizing a high-pressure gas-filled vessel is of crucial importance in petroleum, chemical and biological processing industries. The present study describes a simple model of a slow depressurization process and derives its analytical solution in a recurrence form. This analytical solution is expected to be useful for engineering applications and for the assessment of either detailed numerical simulations or experimental data.

Key words: slow depressurization, gas-filled vessel, mathematical model and solution

1. Introduction

Large steel vessels are often employed in processing industries to contain high-pressure hydrocarbon gaseous mixtures. The pressurized gas may be frequently depressurized to meet the demand of an industrial process. During depressurization, the fluid temperature decreases due to the enthalpy loss with the flow discharging. The decrease of the fluid temperature will provoke heat-transfer processes from the vessel wall to fluid, resulting in the vessel wall temperature decreasing and the fluid temperature recovering. This conjugate heat-transfer process may be quite complicated. A potential safety issue must therefore be taken into consideration *i.e.* whether or not the wall temperature would decline below the ductile-brittle transition temperature of the steel from which the vessel is made, to threaten the integrity of the vessel.

Regarding the depressurization of a high-pressure vessel filled with a certain type of industrial gaseous fluid, there are two problems of particular interest. The first one is the rapid depressurization process or *the blowdown* due to the partial failure of the vessel. The blowdown problem and the related thermal-hydraulic behavior have been extensively investigated in the nuclear industry (*e.g.* Levy [1], Moody [2]). There are also many studies in other industries (*e.g.* Kim [3]).

The second problem is related to a slow and continuous release or discharge of the contained substance because of an isolated valve failure or even under normal discharging operation conditions. This slow depressurization is less dangerous than the rapid one, so that it is also less investigated. Xia *et al.* [4] presented a semi-empirical approximation based on a lumped-parameter model and a data-fitting method. Their approach is basically non-mathematical and gives no error bounds. Xia *et al.* [5] also proposed an analogue in which a constant uniform volumetric heat sink in fluid is employed to simulate the enthalpy loss during slow depressurization of such a vessel. In this second paper they gave an analytical solution of the problem and compared this to a 2-D numerical simulation by the thermal-hydraulic code

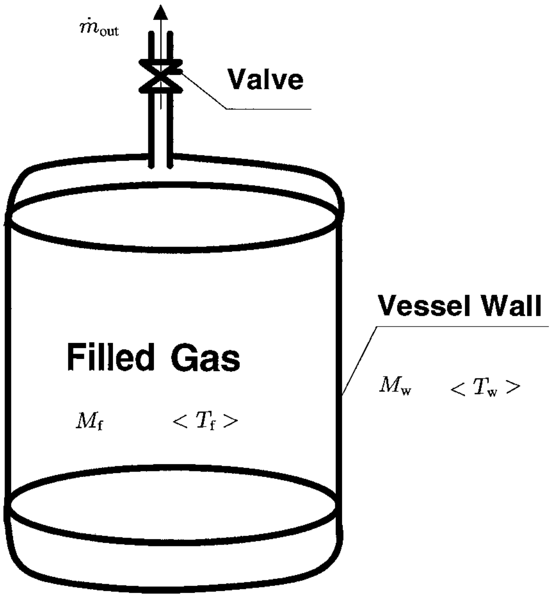

Figure 1. A typical high-pressure gas-filled vessel.

ASTEC [6]. However, their heat-sink analogue is irrelevant to the depressurization due to mass release. Thus, thorough investigation of the potential of analytical solution techniques is still necessary to be carried out.

The present paper employs a simple mathematical model to describe the slow depressurization process of a high-pressure gas-filled vessel and gives its analytical solution. Some aspects related to assess mathematical models and their solutions are discussed. A criterion is derived for the prevention of the ductile-brittle transition of the wall material.

2. Mathematical modeling

Figure 1 depicts a typical gas-filled vessel. The vessel initially contains M_0 kilograms of pressurized gas at temperature T_0 and pressure P_0.[1] The vessel wall is assumed also to be initially at temperature T_0. The depressurization process is started by a release of gas through a valve with a constant discharge flow rate $\dot{m}_{\text{out}}$. Because the main concern is whether or not the wall temperature falls below a critical value, T_{crit}, so that properly modeling the vessel-wall-temperature transient, rather than the details of the fluid flow behavior inside the vessel, is of prevailing importance. The dominant heat-transfer mechanism in this type of gas depressurization is natural convection (Haque *et al.* [7]). Thus, the heat-transfer coefficient between the vessel wall, and the contained fluid is expected to be small. The Biot number, the ratio of the heat transfer to the fluid vs. the heat conduction inside the wall material, is also small. A small Biot number implies that a lumped parameter model can be applied to describe the energy balance of the vessel wall and the fluid.

The basic assumptions in the present study include:

1. The thermodynamic properties $C_{p,w}, C_{p,f}, C_{v,f}$ and ρ_w are constant;

[1] For definition of symbols see nomenclature at the end of the paper.

2. Kinetic and potential energies of the fluid are ignored;
3. The heat-transfer coefficients are all assumed constant;
4. The mass-discharge rate is assumed constant during depressurization.

Although the heat-transfer coefficients may change with time, it s still acceptable to use the *time-averaged* heat-transfer coefficients to describe the average behavior of the conjugate heat transfer if all heat-transfer coefficients do not experience drastic changes. The numerical simulation with the ASTEC code (Xia *et al.* [5]) has shown that in case of the heat-sink driving natural convection, the constant heat-transfer coefficient assumption is fairly good for modeling the temperature transients with a lumped parameter model.

According to the above assumptions, the energy balance equations for both the vessel wall and the fluid can now be written down as:

for the fluid:

$$\frac{\mathrm{d}E_f}{\mathrm{d}t} = \alpha_w A_H(\langle T_w\rangle - \langle T_f\rangle) - \dot{E}_{\mathrm{out}}, \tag{1}$$

for the vessel wall:

$$\frac{\mathrm{d}E_w}{\mathrm{d}t} = -\alpha_w A_H(\langle T_w\rangle - \langle T_f\rangle) + \alpha_\infty A_{H,\infty}(T_\infty - \langle T_w\rangle), \tag{2}$$

where the total energy of the contained fluid, $E_f(t)$, and that of the vessel wall, $E_w(t)$, are defined as below, respectively,

$$E_f(t) = \int_{V_f} C_{v,f}\rho_f(t,\boldsymbol{r})T_f(t,\boldsymbol{r})\,\mathrm{d}V_f \tag{3}$$

and

$$E_w(t) = \int_{V_w} C_{p,w}\rho_w T_w(t,\boldsymbol{r})\,\mathrm{d}V_w. \tag{4}$$

Moreover, the following average temperatures are defined for establishing a closed form of the lumped-parameter model.

$$\langle T_f\rangle \equiv \frac{\int_{V_f} C_{v,f}\rho_f(t,\boldsymbol{r})T_f(t,\boldsymbol{r})\,\mathrm{d}V_f}{\int_{V_f} C_{v,f}\rho_f(t,\boldsymbol{r})\,\mathrm{d}V_f} \tag{5}$$

and

$$\langle T_w\rangle \equiv \frac{\int_{V_w} C_{p,w}\rho_w T_w(t,\boldsymbol{r})\,\mathrm{d}V_w}{\int_{V_w} C_{p,w}\rho_w\,\mathrm{d}V_w}. \tag{6}$$

The fluid mass balance reads:

$$M_f(t) \equiv \int_{V_f} \rho_f(t,\boldsymbol{r})\,\mathrm{d}V_f = \langle\rho_f\rangle V_f = M_{f,0} - \dot{m}_{\mathrm{out}}t, \tag{7}$$

where the average fluid density is a function of time only and defined by

$$\langle \rho_f \rangle \equiv \frac{\int_{V_f} \rho_f(t, \boldsymbol{r})\,\mathrm{d}V_f}{\int_{V_f}\,\mathrm{d}V_f} = \frac{M_f(t)}{V_f}, \tag{8}$$

which yields

$$E_f(t) = C_{v,f} M_f(t) \langle T_f \rangle \quad \text{and} \quad E_w(t) = C_{p,w} M_w \langle T_w \rangle. \tag{9}$$

The summation of Equations (1) and (2) gives:

$$\frac{\mathrm{d}E_f}{\mathrm{d}t} + \frac{\mathrm{d}E_w}{\mathrm{d}t} = \alpha_\infty A_{H,\infty}(T_\infty - \langle T_w \rangle) - \dot{E}_{\text{out}}(t). \tag{10}$$

According to the second assumption, the energy release rate $\dot{E}_{\text{out}}$ can be approximated as,

$$\dot{E}_{\text{out}} = \dot{m}_{\text{out}} C_{p,f} T_{\text{out}}(t), \tag{11}$$

where T_{out} is the temperature at the exit of the valve. The initial conditions are,

$$\langle T_w(0) \rangle = \langle T_f(0) \rangle = T_0 \quad \text{and} \quad T_\infty = \text{Const.} \tag{12}$$

The above mathematical system defined by Equations (1), (2), (7), (9) and (11) is not a closed system yet. Another physical constraint must be added to specify $T_{\text{out}}(t)$. In a lumped-parameter model, we can always correlate $T_{\text{out}}(t)$ with $\langle T_f \rangle$ by imposing a profile function, so that

$$T_{\text{out}}(t) = C_0 \langle T_f \rangle, \tag{13}$$

where constant C_0 takes into account the axial distribution of the fluid temperature. The condition $C_0 = 1$ corresponds to a flat profile.

The present model in principle is valid only for slow depressurization, implying $C_{p,f} T_{\text{out}} \gg \frac{1}{2} u_{\text{out}}^2$. However, the simplification with Equation (13) enables the present model to include the kinetic energy of the released mass if it can be rated to the enthalpy loss as

$$\tfrac{1}{2} u_{\text{out}}^2 = C_K C_{p,f} \langle T_f \rangle, \tag{14}$$

where C_K is a dimensionless coefficient. For a gaseous fluid, $0 < C_K < \frac{1}{2}(C_{p,f}/C_{v,f} - 1)$, which is the sonic velocity limitation. By adding the kinetic energy term $C_K \dot{m}_{\text{out}} C_{p,f} \langle T_f \rangle$ into the energy release rate $\dot{E}_{\text{out}}(t)$ shown in Equation (11), we obtain

$$\dot{E}_{\text{out}}(t) = C_{\text{out}} \dot{m}_{\text{out}} C_{p,f} \langle T_f \rangle, \tag{15}$$

where $C_{\text{out}} = C_K + C_0$ should be slightly bigger than one. If the slow depressurization is emphasized, $C_{\text{out}} = 1$ is not only physically reasonable, but also mathematically convenient without loss of generality. The model described above is called the direct mass-discharge model.

3. Analytical solution

An analytical solution for the above model exists under the imposed physical assumptions. By substituting Equations (7), (9) and (15) in Equations (1), (2) and (12) and making them dimensionless, one obtains the following governing equations:

$$\frac{\mathrm{d}\Theta_w}{\mathrm{d}\tau} = -\left(\frac{\tau_m}{\tau_w} + \frac{\tau_m}{\tau_\infty}\right)\Theta_w + \frac{\tau_m}{\tau_w}\Theta_f + \frac{\tau_m}{\tau_\infty}\Theta_\infty, \tag{16}$$

$$\frac{\mathrm{d}\Theta_f}{\mathrm{d}\tau} = \frac{\tau_m}{\tau_f}\frac{\Theta_w}{1-\tau} - \left(\frac{\tau_m}{\tau_f} + \frac{C_{p,f}}{C_{v,f}}C_{\mathrm{out}} - 1\right)\frac{\Theta_f}{1-\tau}, \tag{17}$$

$$\tau = 0, \quad \Theta_w(0) = \Theta_f(0) = 1; \qquad \Theta_\infty = \frac{T_\infty}{T_0} = \mathrm{Const}, \tag{18}$$

where the time constants are defined as:

$$\tau_m = \frac{M_{f,0}}{\dot{m}_{\mathrm{out}}}; \qquad \tau_w = \frac{M_w C_{p,w}}{\alpha_w A_H}; \qquad \tau_f = \frac{M_{f,0} C_{v,f}}{\alpha_w A_H}; \qquad \tau_\infty = \frac{M_w C_{p,w}}{\alpha_\infty A_{H,\infty}}, \tag{19}$$

and the definitions of all dimensionless parameters are:

$$\Theta_w = \frac{\langle T_w \rangle}{T_0}; \qquad \Theta_f = \frac{\langle T_f \rangle}{T_0}; \qquad \tau = \frac{t}{\tau_m}. \tag{20}$$

The above time constants have clear physical meanings: τ_m is the mass-discharge time constant, τ_w and τ_∞ are the vessel-wall thermal-transition time constants relevant to the heat exchanges of *the wall to the fluid* and *the environment to the wall*, respectively, τ_f is the fluid thermal-transition time constant. These time constants actually control the thermal-transition process of the system.

The case where we have an adiabatic boundary at the outside surface of the vessel wall is the most important situation. Otherwise, the heat transferred from the environment will balance the energy loss due to the discharge of the contained mass. The present study concentrates on this case, which yields

$$\frac{\mathrm{d}\Theta_w}{\mathrm{d}\tau} = -\frac{\tau_m}{\tau_w}\Theta_w + \frac{\tau_m}{\tau_w}\Theta_f, \tag{21}$$

$$\frac{\mathrm{d}\Theta_f}{\mathrm{d}\tau} = \frac{\tau_m}{\tau_f}\frac{\Theta_w}{1-\tau} - \eta\tau_m\frac{\Theta_f}{1-\tau}, \tag{22}$$

$$\tau = 0, \qquad \Theta_w(0) = \Theta_f(0) = 1, \tag{23}$$

where a constant η is defined as,

$$\eta = \frac{1}{\tau_f} + \frac{\frac{C_{p,f}}{C_{v,f}}C_{\mathrm{out}} - 1}{\tau_m}. \tag{24}$$

We can decouple the above equations by further differentiating them and obtain a set of equivalent second-order linear equations:

for the vessel wall:

$$\frac{\mathrm{d}^2\Theta_w}{\mathrm{d}\tau^2} + \frac{\tau_m}{\tau_w}\left(1 + \frac{\eta\tau_w}{1-\tau}\right)\frac{\mathrm{d}\Theta_w}{\mathrm{d}\tau} + \frac{\tau_m}{\tau_w}\frac{\frac{C_{p,f}}{C_{v,f}}C_{\text{out}} - 1}{1-\tau}\Theta_w = 0,$$
$$\tau = 0, \quad \Theta_w(0) = 1; \qquad \frac{\mathrm{d}\Theta_w}{\mathrm{d}\tau}(0) = 0, \tag{25}$$

and for the fluid:

$$\frac{\mathrm{d}^2\Theta_f}{\mathrm{d}\tau^2} + \frac{\tau_m}{\tau_w}\left(1 + \frac{\eta\tau_w - \frac{\tau_w}{\tau_m}}{1-\tau}\right)\frac{\mathrm{d}\Theta_f}{\mathrm{d}\tau} + \frac{\tau_m}{\tau_w}\frac{\frac{C_{p,f}}{C_{v,f}}C_{\text{out}} - 1}{1-\tau}\Theta_f = 0,$$
$$\tau = 0, \quad \Theta_f(0) = 1; \qquad \frac{\mathrm{d}\Theta_f}{\mathrm{d}\tau}(0) = -\left(\frac{C_{p,f}}{C_{v,f}}C_{\text{out}} - 1\right). \tag{26}$$

If one of these is solved, the solution of the rest can be obtained from the original equations given by Equations (21) to (23).

The present study solves Equation (25) for the wall temperature first. Equation (25) has one pole singularity at $\tau = 1$ in the coefficient functions. According to the general theory of linear ordinary differential equations (e.g. Nayfeh [8]), the trial solution of Equation (25) may be expressed as

$$\Theta_w = (1-\tau)^\sigma \sum_{n=0}^{\infty} a_n (1-\tau)^n = \sum_{n=0}^{\infty} a_n (1-\tau)^{\sigma+n}. \tag{27}$$

By substituting Equation (27) and its derivatives in Equation (25), we can have,

$$\sum_{n=0}^{\infty} a_n(\sigma+n)(\sigma+n-1-\eta\tau_m)(1-\tau)^{\sigma+n-2}$$
$$-\sum_{n=0}^{\infty}\frac{\tau_m}{\tau_w}\left(\sigma+n-\left(\frac{C_{p,f}}{C_{v,f}}C_{\text{out}}-1\right)\right)a_n(1-\tau)^{\sigma+n-1} = 0. \tag{28}$$

The leading term is the one which corresponds to $(1-\tau)^{\sigma-2}$ when $n=0$. We can eliminate the leading term by selecting the unknown constant σ in its coefficient, *i.e.* by imposing,

$$a_0\sigma(\sigma - 1 - \eta\tau_m) = 0. \tag{29}$$

Then, let the coefficients of all terms at different order of $(1-\tau)$ be equal to zero, which yields:

$$a_{n+1} = \frac{\frac{\tau_m}{\tau_w}\left(\sigma+n-\left(\frac{C_{p,f}}{C_{v,f}}C_{\text{out}}-1\right)\right)}{(\sigma+n+1)(\sigma+n-\eta\tau_m)}a_n; \quad n = 0, 1, \ldots, \infty. \tag{30}$$

It is clear that $a_0 \neq 0$, otherwise the solution would become trivial. Thus, we can either set $\sigma = 0$ or $\sigma - 1 - \eta\tau_m = 0$ in Equation (29). These two series for $\sigma = 0$ and $\sigma = 1 + \eta\tau_m$ can be written as

$$a_{n+1} = \frac{\frac{\tau_m}{\tau_w}\left(n-\left(\frac{C_{p,f}}{C_{v,f}}C_{\text{out}}-1\right)\right)}{(n+1)(n-\eta\tau_m)}a_n; \quad n = 0, 1, \ldots, \infty \tag{31}$$

and

$$b_{n+1} = \frac{\frac{\tau_m}{\tau_w}\left(1 + \eta\tau_m + n - \left(\frac{C_{p,f}}{C_{v,f}} C_{\text{out}} - 1\right)\right)}{(\eta\tau_m + n + 2)(n + 1)} b_n; \quad n = 0, 1, \ldots, \infty, \tag{32}$$

respectively. The absolute convergence does hold true for both options in the region of interest which is $\tau \in [0, 1]$. Therefore, they correspond to the two independent solutions of Equation (25) and can be expressed as,

$$\Theta_{w,1} = \sum_{n=0}^{\infty} a_n (1 - \tau)^n \quad \text{and} \quad \Theta_{w,2} = \sum_{n=0}^{\infty} b_n (1 - \tau)^{\eta\tau_m + n + 1}, \tag{33}$$

in which only two constants a_0 and b_0 need to be specified.

The solution of the vessel-wall temperature can be derived from,

$$\Theta_w = C_1 \Theta_{w,1} + C_2 \Theta_{w,2}, \tag{34}$$

where C_1 and C_2 are two constants related to the initial conditions of Equation (23). Because a_0 and b_0 are also non-vanishing, they can be combined with C_1 and C_2. Therefore, setting $a_0 = 1$ and $b_0 = 1$ we will not incur loss of generality.

The boundary conditions result in:

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \Theta_{w,1}(0) & \Theta_{w,2}(0) \\ \frac{\mathrm{d}\Theta_{w,1}}{\mathrm{d}\tau}(0) & \frac{\mathrm{d}\Theta_{w,2}}{\mathrm{d}\tau}(0) \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{35}$$

so that

$$\Theta_w(\tau) = (\Theta_{w,1}(\tau), \Theta_{w,2}(\tau)) \cdot \begin{pmatrix} \Theta_{w,1}(0) & \Theta_{w,2}(0) \\ \frac{\mathrm{d}\Theta_{w,1}}{\mathrm{d}\tau}(0) & \frac{\mathrm{d}\Theta_{w,2}}{\mathrm{d}\tau}(0) \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{36}$$

Once $\Theta_w(\tau)$ has been solved analytically, the solution of Equation (26) for the fluid temperature can be obtained simply from Equation (21):

$$\Theta_f = \Theta_w + \frac{\tau_w}{\tau_m} \frac{\mathrm{d}\Theta_w}{\mathrm{d}\tau}. \tag{37}$$

It is also possible to extend the solution to that for a more general system defined by Equations (16) to (18). By combining Equation (16) with Equation (17), we can derive a general second-order linear equation either for the vessel wall or for the fluid as,

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}\tau^2} + \left(\beta_1 + \frac{\beta_2}{1 - \tau}\right) \frac{\mathrm{d}Y}{\mathrm{d}\tau} + \frac{\gamma_1}{1 - \tau} Y + \frac{\gamma_2}{1 - \tau} = 0, \tag{38}$$

where $\beta_{1,2}$ and $\gamma_{1,2}$ are some known constants composed by the system parameters and the physical properties. The theory of linear ordinary differential equations ensures that the general solution of the above equation $Y(\tau)$ is linearly composed by the general solution of the corresponding homogeneous equation $\overline{Y}(\tau)$ and a particular solution of the original non-homogeneous equation $Y^*(\tau)$. Fortunately, the original equation leads to

$$Y^* = -\frac{\gamma_2}{\gamma_1}. \tag{39}$$

By letting $Y = \overline{Y} - \gamma_2/\gamma_1$ and substituting it in Equation (38), we obtain

$$\frac{\mathrm{d}^2\overline{Y}}{\mathrm{d}\tau^2} + \left(\beta_1 + \frac{\beta_2}{1-\tau}\right)\frac{\mathrm{d}\overline{Y}}{\mathrm{d}\tau} + \frac{\gamma_1}{1-\tau}\overline{Y} = 0. \tag{40}$$

Following the solution procedure discussed before, we can also find the solution for $\overline{Y}$. As we mentioned previously, the general solution is less important than the one we have thoroughly depicted; thus the details are omitted.

4. Discussions of the model and the solution

From the structure of the solutions, the present study gives rise to two main interesting points which can be used to define the minimum possible wall temperature and maximum possible fluid temperature.

Firstly, the solution for the wall temperature can be proven to be a monotonically declining function by the following transform:

$$\Theta_w(\tau) = \beta \exp\left(\int_0^\tau p(\tau)\,\mathrm{d}\tau\right), \tag{41}$$

where β is a trial constant and $p(\tau)$ a trial function. By substituting the above trial solution in the original differential Equation (25), we obtain

$$\left(\frac{\mathrm{d}p(\tau)}{\mathrm{d}\tau} + p^2(\tau) + g_1(\tau)p(\tau) + g_2(\tau)\right)\Theta_w(\tau) = 0, \tag{42}$$

where

$$g_1(\tau) = \frac{\tau_m}{\tau_w}\left(1 + \frac{\eta\tau_w}{1-\tau}\right) \quad \text{and} \quad g_2(\tau) = \frac{\tau_m}{\tau_w}\frac{\frac{C_{p,f}}{C_{v,f}}C_{\mathrm{out}} - 1}{1-\tau}. \tag{43}$$

Because $\Theta_w(\tau) = 0$ is only a trivial solution, then

$$\frac{\mathrm{d}p(\tau)}{\mathrm{d}\tau} + p^2(\tau) + g_1(\tau)p(\tau) + g_2(\tau) = 0, \tag{44}$$

with converted initial conditions $\beta = 1$ and $p(0) = 0$.

This functional transform can help us to understand some characteristics of the solution without solving the derived equation. Generally, to find an analytical solution for the above nonlinear system is more difficult than for the original linear system, if not impossible.

That the characteristics of $\Theta_w(\tau)$ are monotonically declining can be ensured, once it has been proved that the trial function $p(\tau)$ is always negative. In order to do that, the above characteristic equation is re-written as:

$$\frac{\mathrm{d}p(\tau)}{\mathrm{d}\tau} = -(p(\tau) - \lambda_1(\tau))(p(\tau) - \lambda_2(\tau)), \tag{45}$$

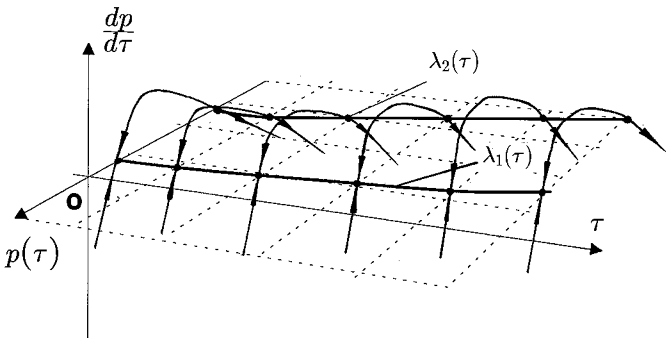

Figure 2. $(\mathrm{d}p/\mathrm{d}\tau) - p - \tau$ phase diagram of the characteristics of the wall temperature Θ_w and the fluid temperature Θ_f with $\lambda_2 < \lambda_1 < 0$.

where $\lambda_1(\tau)$ and $\lambda_2(\tau)$ are the *fixed points* of the above equation and obtained from

$$\lambda^2(\tau) + g_1(\tau)\lambda(\tau) + g_2(\tau) = 0. \tag{46}$$

We can easily prove that the two roots of the above equation, $\lambda_{1,2}(\tau)$, are negative. The characteristics of $p(\tau)$ are topologically shown in a $(\mathrm{d}p/\mathrm{d}\tau) - p - \tau$ phase diagram (Figure 2) which indicates that $\lambda_1(\tau)$ is an *attractor* and $\lambda_2(\tau)$ an *expeller*. From this phase diagram, we can verify that, once $p(\tau)$ becomes negative, it will remain so. Taking into account the converted initial condition $p(0) = 0$ and the characteristics of $\mathrm{d}p(\tau)/\mathrm{d}\tau$, we can prove that $p(\tau)$ is always negative and $\Theta_w(\tau)$ is a monotonically declining function when $\tau > 0$.

The monotonically declining characteristics of the fluid temperature are also preserved if $\mathrm{d}\Theta_f(0)/\mathrm{d}\tau$ is absolutely negative. Actually, we can prove that at any time τ, both fixed points of the corresponding characteristic equation similar to Equation (45) always keep the same sign, so that $p(\tau)$ will remain negative with a converted initial condition $p(0) < 0$. This obviously holds true because $g_2(\tau)$ is positive definite. Three typical possible cases are shown in the $(\mathrm{d}p/\mathrm{d}\tau) - p - \tau$ phase diagram of Figures 2, 3 and 4. Moreover, from Equation (37), we can verify that the upper limit of Θ_f is bounded by the corresponding Θ_w at an arbitrary time $0 < \tau < 1$.

Both the wall and the fluid temperatures will drop to a minimum when $\tau = 1$. By substituting $\tau = 1$ in Equations (34) to (36), we obtain:

$$\Theta_{w,\min} = \Theta_w(1) = C_1 = f\left(\frac{\tau_m}{\tau_w}, \frac{\tau_m}{\tau_f}, \frac{C_{p,f}}{C_{v,f}}C_{\text{out}}\right), \tag{47}$$

and for the fluid temperature

$$\Theta_{f,\min} = \Theta_w(1) + \frac{\tau_w}{\tau_m}\frac{\mathrm{d}\Theta_w(1)}{\mathrm{d}\tau} = C_1\left(1 - \frac{\frac{C_{p,f}}{C_{v,f}}C_{\text{out}} - 1}{\eta\tau_m}\right). \tag{48}$$

The monotonically declining characteristics of Θ_w ensure $C_1 < 1$. The relative difference of both minimum temperatures can therefore be written as

$$R_{\Theta_{\min}} = \frac{\Delta\Theta_{\min}}{\Theta_{w,\min}} \equiv \frac{\Theta_{w,\min} - \Theta_{f,\min}}{\Theta_{w,\min}} = \left(\frac{\frac{C_{p,f}}{C_{v,f}}C_{\text{out}} - 1}{\frac{\tau_m}{\tau_f} + \frac{C_{p,f}}{C_{v,f}}C_{\text{out}} - 1}\right) < 1. \tag{49}$$

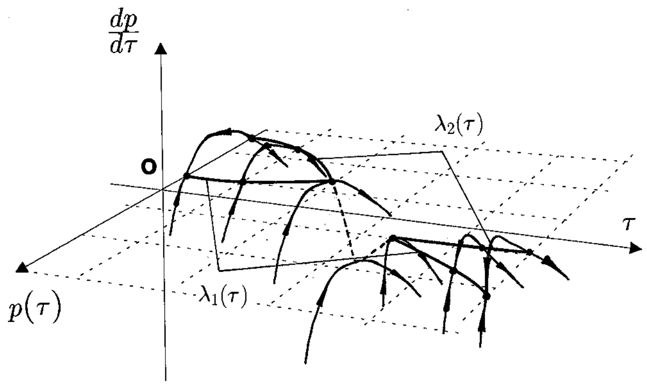

Figure 3. $(\mathrm{d}p/\mathrm{d}\tau) - p - \tau$ phase diagram of the characteristics of the fluid temperature Θ_f, $\lambda_{1,2}$ may change signs simultaneously.

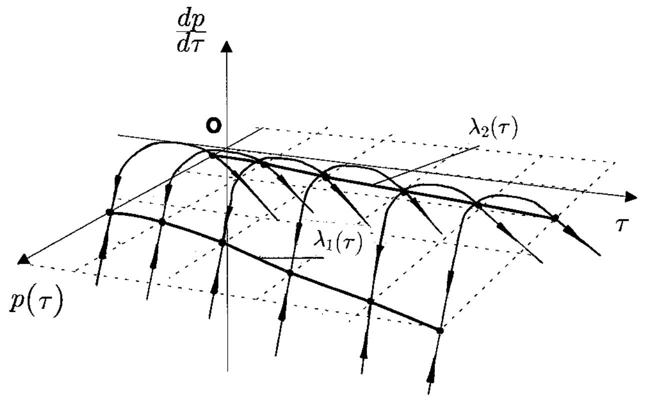

Figure 4. $(\mathrm{d}p/\mathrm{d}\tau) - p - \tau$ phase diagram of the characteristics of the fluid temperature Θ_f, $\lambda_{1,2} > 0$.

The last equation ensures that $\Delta\Theta_{\min}$ will tend to zero if $(\tau_m/\tau_f) \gg 1$, which physically corresponds to $\alpha_w A_H \gg \dot{m}_{\text{out}} C_{v,f}$. For most of engineering gases, $(C_{p,f}/C_{v,f})C_{\text{out}}$ is only slightly larger than one, but less than two; $(\tau_m/\tau_f) \approx 5$ could result in $R_{\Theta_{\min}} < 0.1$. The analytical results obtained from Equations (35), (48) and (49) have shown that C_1 decreases monotonously and $\Theta_{f,\min}$ increases monotonously either with α_w increasing or with $\dot{m}_{\text{out}}$ decreasing, respectively. Because $(\partial R_{\Theta_{\min}}/\partial C_{\text{out}}) > 0$ is preserved from Equation (49), an increase of C_{out} leads to a bigger difference between the wall and the fluid temperatures. Physically, it coincides with the fact that the faster the energy-release rate is, the less heat is transferred to the fluid.

Secondly, the monotonic variations of both wall and fluid temperatures and their minimum values actually indicate the existence of a minimum possible wall temperature $\overline{\Theta_w(\tau)}_{\min}$ and a maximum possible fluid temperature $\overline{\Theta_f(\tau)}_{\max}$. They correspond to the physical condition $(\tau_m/\tau_f) \gg 1$. From Equation (49) we can directly obtain an asymptotic approximation as

$$\Theta(\tau) = \overline{\Theta_w(\tau)}_{\min} \approx \overline{\Theta_f(\tau)}_{\max}. \tag{50}$$

A combination of Equations (21) and (22) leads to

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} + \frac{\left(\frac{C_{p,f}}{C_{v,f}}C_{\text{out}} - 1\right)}{\frac{\tau_w}{\tau_f} + 1 - \tau}\Theta = 0. \tag{51}$$

The integration of the above equation yields a limiting value for the minimum possible wall temperature and the maximum possible fluid temperature, whence

$$\Theta(\tau) = \overline{\Theta_w(\tau)}_{\text{min}} \approx \overline{\Theta_f(\tau)}_{\text{max}} \approx \left(1 - \frac{\tau}{1 + \frac{\tau_w}{\tau_f}}\right)^{(C_{p,f}/C_{v,g})C_{\text{out}}-1}. \tag{52}$$

The definitions for τ_w and τ_f yield $(\tau_w/\tau_f) = (M_w C_{p,w}/M_{f,0}C_{v,f})$, so that the heat-transfer coefficient which is an average parameter in the present study is not involved in such extreme cases. The equation itself is a declining function, as expected. By setting $\tau = 1$ into this equation, we can derive a general safety criterion, which is naturally defined as

$$T_0\left(1 - \frac{1}{1 + \frac{\tau_w}{\tau_f}}\right)^{(C_{p,f}/C_{v,f})C_{\text{out}}-1} > T_{\text{crit}}. \tag{53}$$

Again, this criterion is related neither to the heat-transfer coefficient, nor to the mass-discharge rate, but to the ratio of thermal capacity of the vessel wall vs. the fluid and to the thermodynamic properties of the fluid. How these parameters control the depressurization process has become clear.

Figure 5 shows the effect of heat-transfer coefficient to the vessel wall and the fluid temperatures with $C_{\text{out}} = 1$. As expected, the larger the heat-transfer coefficient, the lower the vessel-wall temperature and the higher the fluid temperature. Figure 6 describes the effect of the mass-discharge rate with $C_{\text{out}} = 1$. Again as expected, the faster the discharge rate, the higher the vessel-wall temperature and the lower the fluid temperature. Figure 7 is a 3-D diagram in $\tau - (\tau_w/\tau_f) - \Theta_w$ to show how the safety criterion should be applied.

The model discussed in this paper is only an energy-governed system, because the mass balance is pre-defined. The total energy conservation should be additionally integrated and used as a criterion to justify solutions obtained either numerically or analytically. Because the present study employs a lumped-parameter model, the integral form of energy conservation can be readily processed by the following simple quadratures:

$$\begin{aligned}\Delta E_{R,m} \equiv\ & \frac{\tau_w}{\tau_f}(1 - \Theta_w) + (1 - (1 - \tau)\Theta_f) \\ & + \int_0^\tau \frac{\tau_m \tau_w}{\tau_\infty \tau_f}(\Theta_\infty - \Theta_w)\,\mathrm{d}\tau - \frac{C_{p,f}}{C_{v,f}}C_{\text{out}}\int_0^\tau \Theta_f\,\mathrm{d}\tau < \varepsilon_m,\end{aligned} \tag{54}$$

where ε_m is the imposed tolerance for energy conservation.

To have a proper analytical solution of the slow depressurization of a gas-filled vessel due to mass release or discharge is expected to have an additional importance for code simulation assessment. It has been recognized that an acceptable numerical solution must be justified by the combination of (Wulff [9]):

(1) estimates of computing error bounds;

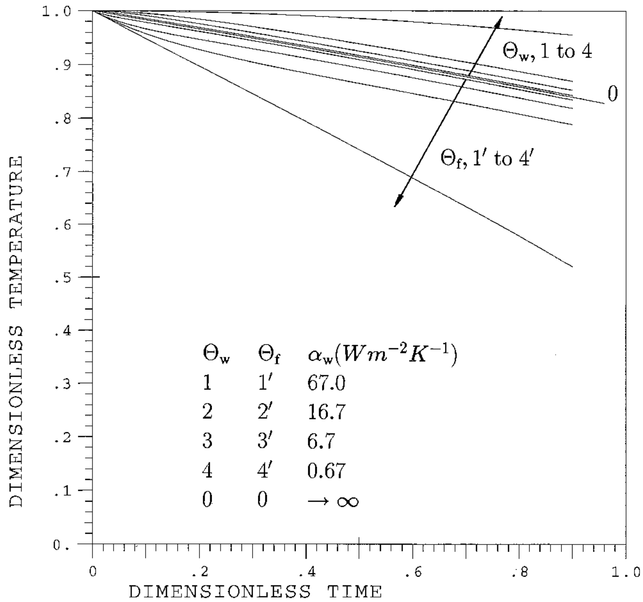

Figure 5. Effect of heat-transfer coefficient; $(\tau_w/\tau_f) = 2{\cdot}1122$, $(C_{p,f}/C_{v,f}) = 1{\cdot}5119$, $\tau_m = 29347{\cdot}1782$ s, $\dot{m}_{\text{out}} = 2{\cdot}1033\,\text{kgs}^{-1}$. From the line 0 upwards, lines are 1, 2, 3, 4 for Θ_w; and downwards, lines are $1', 2', 3', 4'$ for Θ_f, with respect to different α_w in pairs.

(2) comparisons with analytical solutions; and
(3) substitution of the results into the original equations.

The comparisons between the numerical calculations and the well-established experimental data are important to evaluate the mathematical modeling of the physical problem, but are never sufficient to justify numerical simulations and unjustified approximations, because such comparisons fail to reveal the compensation of modeling deficiencies through computing errors.

Conclusions

The present study describes a simple model for slow depressurization of a typical high-pressure gas-filled vessel. Its analytical solution has been derived and the validity was discussed. The present study of a direct mass-discharge model is believed to exhibit a convincing picture of such a special engineering problem.

A conservative safety criterion based on comprehensive investigation of the analytical solution has been established. For the sample system, a reasonable value of the heat-transfer coefficient α_w should be below 20 $\text{Wm}^{-2}\text{K}^{-1}$, according to the definition of the present paper. Otherwise, the temperature difference between the wall and the fluid would become negligible and Equation (52) will not only function as a conservative approximation, but also will depict the average temperature transients.

We hope that the present study has increased our understanding of the physical phenomenon of slow depressurization of a gas-filled vessel. The analytical solutions and the safety criterion obtained are useful not only in engineering applications, but also in assessing numerical simulations or even experimental data.

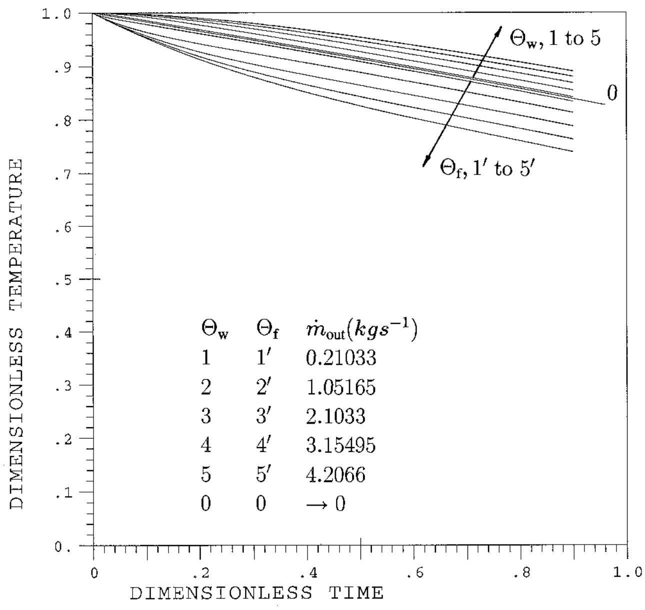

Figure 6. Effect of mass-discharge rate; $(C_{p,f}/C_{v,f}) = 1{\cdot}5119$, $\tau_w = 13102{\cdot}92$ s, $\alpha_w = 6{\cdot}7\,\mathrm{Wm^{-2}K^{-1}}$. From the line 0 upwards, lines are 1, 2, 3, 4, 5 for Θ_w; and downwards, lines are $1', 2', 3', 4', 5'$ for Θ_f, with respect to different $\dot{m}_{\text{out}}$ in pairs.

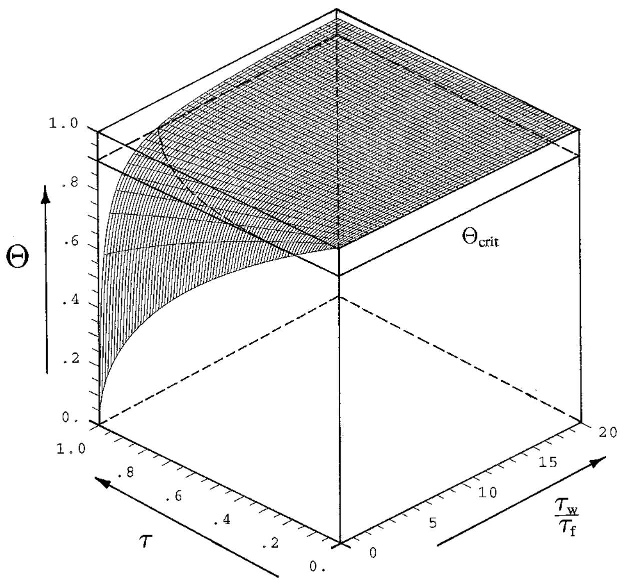

Figure 7. Safety criterion for minimum possible wall temperature $\overline{\Theta_w(\tau)}_{\text{min}}$.

Nomenclature

Parameter and variable

A_H heat transfer area between the wall and the fluid, m^2
C_K coefficient defined by Equation (14) to correlate the kinetic energy with the flow enthalpy, dimensionless
C_p specific heat at constant pressure, J kg^{-1} K^{-1}

C_v	specific heat at constant volume, J kg^{-1} K^{-1}
C_0	coefficient to correlate T_{out} with $\langle T_f \rangle$, dimensionless
E	energy, J
$\dot{E}_{\text{out}}$	energy release rate due to mass discharge, J s^{-1}
M	mass, kg
$\dot{m}_{\text{out}}$	mass discharge rate, kg s^{-1}
p	trial function defined by Equation (41), dimensionless
P	system pressure, N m^{-2}
$\mathbf{r}$	position vector in space, m
R_Θ	relative dimensionless temperature difference defined by Equation (49)
T	temperature, K
t	time, s
u	fluid velocity, m s^{-1}
V	volume, m^3
Y	dimensionless temperature of a general system defined by Equation (38)
α	heat transfer coefficient, W m^{-2} K^{-1}
λ	roots of the characteristic Equation (46)
ρ	density, kg m^{-3}
Θ	dimensionless temperature
τ	dimensionless time (or time constant, s)
ε_m	convergence tolerance of energy conservation applied in Equation (54)

Subscript and symbol

crit	criterion
f	fluid
H	heat transfer surface
m	discharging mass flow
max	maximum
min	minimum
w	wall
out	flowing out
0	initial value
∞	interface to the environment
Δ	difference
$\langle\psi\rangle$	average value of function $\psi, \psi = T_w$, or T_f, or ρ_f
$\overline{\psi}$	asymptotic value of function $\psi, \psi = \Theta_w$, or Θ_f, or Y

Acknowledgements

The authors are gratefully indebted to H.M. Friess of Nuclear Engineering Laboratory of Swiss Federal Institute of Technology for numerous discussions about mathematical disciplines and skills in many aspects of processing the analytical solutions. The authors also greatly appreciate all efforts of the editor-in-chief and the reviewers for their valuable comments and suggestions regarding the quality improvement of this paper.

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